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Forcing matching numbers of fullerene graphs[☆]

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ABSTRACT

The forcing number or the degree of freedom of a perfect matching M of a graph G is the cardinality of the smallest subset of M that is contained in no other perfect matchings of G. In this paper we show that the forcing numbers of perfect matchings in a fullerene graph are not less than 3 by applying the 2-extendability and cyclic edge-connectivity 5 of fullerene graphs obtained recently, and Kotzig's classical result about unique perfect matching as well. This lower bound can be achieved by infinitely many fullerene graphs.

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1. Introduction

A *fullerene graph* is a 3-regular (cubic) connected plane graph (or spherical map) with only pentagonal faces and hexagonal faces. By Euler's formula, a simple argument shows that a fullerene graph must have only 12 pentagonal faces. As a class of trivalent polyhedra, fullerene graphs have been studied in mathematics for a long time [9,10]; for example, the dodecahedron is the fullerene graph with 20 vertices. The structures of fullerene graphs are particularly interested to mathematicians [4,11]. From chemical point of view, fullerene graphs are the molecular graphs of fullerenes; for example, the famous molecule C_{60} is one of fullerenes, discovered by Kroto et al. [16] in 1985.

A set of independent edges of a fullerene graph F is called a *matching* of F. A *perfect matching* (or Kekulé structure) M of F is a matching such that every vertex is incident with exactly one edge in M. A *forcing set* for a perfect matching M of a fullerene graph F is a subset S of M such that S is contained in the unique perfect matching of F. The minimum cardinality of forcing sets of M is called the *forcing number* (or *degree of freedom*) *of* M, and is denoted by $f(F; M)$. Kekulé structures play a key role in the molecule resonance energy and aromaticity of organic molecules. For a recent survey, refer to [22]. For a fullerene, Kekulé structures do not contribute to its molecule resonance energy equally [7]. Recent works [23,25–27] show that the Kekulé structures of fullerenes with a large degree of freedom are more important than those of fullerenes with a smaller degree of freedom in resonance theory. The *spectrum* of forcing numbers of perfect matchings of F is defined as $Spec(F) := \{f(F; M) \mid M \text{ is a perfect matching of } F\}$. The *forcing number* of a fullerene graph F is the minimum number of forcing numbers of all perfect matchings of F, denoted by $f(F)$.

The concept of forcing number of graphs was originally introduced for benzenoid systems by Harary et al. [12]. The same idea appeared in an earlier paper [13] of Klein and Randić by the name "innate degree of freedom". The benzenoid systems

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with minimum forcing number 1 had been investigated in [19,29–31], as well as plane bipartite graphs [33]. Recently, the minimum forcing numbers of bipartite graphs [1,2] have been extensively studied, especially for the stop signs [18], square grid [20], torus and hypercube [1,14,24]. To determine whether a set is a minimum forcing set of a perfect matching of a bipartite graph with maximum degree 3 is NP-complete [1]. Wang et al. [28] gave a linear-time algorithm for computing the minimum forcing number of a toroidal polyhex according to its geometric structure.

It is interesting to consider the minimum forcing numbers of general fullerene graphs which are non-bipartite. As we know, little work has been done on the minimum forcing numbers of non-bipartite graphs. Vukičević, Kroto and Randić [26,27] computed the forcing numbers of perfect matchings of C_{60} and their result implies that $Spec(C_{60}) = \{5, 6, 7, 8, 9, 10\}$ and hence $f(C_{60}) = 5$. In this paper, we consider the minimum forcing numbers of general fullerene graphs F. Applying the 2-extendability and cyclic edge-connectivity 5 of fullerene graphs, which were obtained recently, and also combining Kotzig's classical result about graphs with a unique perfect matching, we obtain the following main result:

Theorem 1.1. *Let F be a fullerene graph. Then $f(F) \geq 3$.*

This bound is achieved by infinitely many fullerene graphs, including all fullerene graphs with a non-trivial cyclic 5-edge-cut.

All undefined concepts and notations in this paper can be found in [3].

2. Proof of the main result

A connected graph G with at least six vertices is said to be 2-*extendable* if any two disjoint edges of G belong to a perfect matching.

Theorem 2.1 ([32]). *Every fullerene graph is* 2*-extendable.*

A graph G is said to be *cyclically k-edge-connected* if at least k edges must be removed to disconnect G into two components, each containing a cycle. Such a set of k edges is called a *cyclic k-edge-cut* and it is *trivial* if at least one of the two components is a single cycle of length k. The *cyclic edge-connectivity* of G, denoted by $c\lambda(G)$, is the maximum integer k such that G is cyclically k-edge-connected.

Theorem 2.2 ([6,21]). *Let F be a fullerene graph. Then $c\lambda(F) = 5$.*

By Theorem 2.2 and the 3-connectivity of F, a cycle of F has length 5 or 6 if and only if it bounds a face of F. A *pentacap* is a graph consisting of 6 pentagons, as shown in Fig. 10 (left).

Theorem 2.3 ([17]). *Let F be a fullerene admitting a nontrivial cyclic* 5*-edge-cut. Then F contains a pentacap, and more precisely, F contains two disjoint antipodal pentacaps.*

Let F be a fullerene graph admitting a nontrivial cyclic 5-edge-cut S. In fact, in the proof of Theorem 2.3, Kutnar and Marušič [17] showed that removing edges in S separates F into two 2-connected subgraphs, each of which has a face of size 10 on which the 2-degree vertices and 3-degree vertices appear alternately along some direction, and the edges in S are not incident with any pentagon of F. We summarize the above properties in the following proposition:

Proposition 2.4 ([17]). *Let F be a fullerene graph admitting a non-trivial cyclic* 5*-edge-cut S. Then,*

(1) *S does not consist of the edges incident with a pentagon of F;*
(2) *every component of $F - S$ has a face of size* 10 *on which the* 2*-degree vertices and* 3*-degree vertices appear alternately along some direction.*

A *bridge* of a graph is an edge whose deletion increases the number of components.

Theorem 2.5 ([15]). *Let G be a connected graph with a unique perfect matching. Then G has a bridge belonging to the perfect matching.*

Suppose that G is a subgraph of F. We use $F - G$ to denote the graph $F - V(G)$. A connected subgraph G of F is *unforcing* if $F - G$ has no vertices of degree 1 and every inner face of G is also a face of F. Define G^* to be the graph arising from an unforcing subgraph G of F as follows. Substitute each vertex of G by a triangle such that two triangles of G^* share an edge if and only if the corresponding vertices in G are adjacent. Since every inner face of G is also a face of F, every inner face of G^* is a triangle. In fact, G is an inner dual of G^*. Note that G^* is not always a subgraph of the dual graph of F since G^* may have some vertices on its boundary corresponding to the same face of F. Denote the diameter of G^* by $diam(G^*)$. For example, the subgraph G induced by thick edges and G^* are shown in Fig. 1. Every edge e' of G^* crosses an edge e in F, called the *corresponding edge* of e'.

Lemma 2.6. *Let F be a fullerene graph. Let G be an unforcing subgraph of F and G^* be defined as above. We have the following statements.*

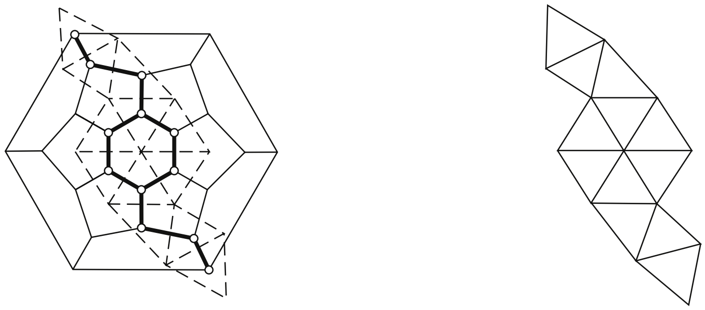

Fig. 1. A subgraph G induced by thick edges (left) and G^* (right).

(1) *If $diam(G^*) \leq 3$, $F - G$ is 2-connected;*
(2) *If $diam(G^*) = 4$, either $F - G$ is 2-connected or $F - G$ is connected and has a bridge e. When the latter holds, let e be the common edge of faces f_0 and f_1 of F. Then G^* has two vertices f_0^* and f_1^* corresponding to f_0 and f_1 such that a shortest path P of G^* connecting f_0^* and f_1^* has length 4 and, together with e, the edges corresponding to edges in P form a cyclic 5-edge-cut of F.*

Proof. First, we show that $F - G$ is connected if $diam(G^*) \leq 4$. If not, then G separates $F - G$. Since $F - G$ has no vertices of degree 1, each component of $F - G$ has a cycle. Since every inner face of G is also a face of F, two components of $F - G$ lie in the infinite face of G. Let $S \subset E(G)$ be a minimum edge-cut separating $F - G$, which of course is a cyclic edge-cut. By the definition of G^* and the minimality of S, the edges in G^* corresponding to edges in S form a shortest path P. So $|S| \leq diam(G^*) \leq 4$, contradicting that $c\lambda(F) = 5$.

Since both $F - G$ and G are connected, $F - G$ has only one face f other than pentagons and hexagons, and its boundary ∂f is a closed walk. Assume that $\partial f = v_0e_0v_1e_1v_2e_2 \cdots v_me_mv_0$. If ∂f is a cycle, then $F - G$ is clearly 2-connected.

So, in the following, suppose that ∂f is not a cycle. Then there are $0 \leq i \neq j \leq m$ such that $e_i = e_j$. So e_i is a bridge of $F - G$. Now assume that $e_i = f_0 \cap f_1$. By the 3-connectivity of F, $f_0 \neq f_1$. Let f_0^* and f_1^* be the vertices of G^* corresponding to f_0 and f_1, respectively. Let $P := f_1^* f_2^* \cdots f_r^* f_0^*$ be a shortest path of G^* connecting f_1^* and f_0^*. Since P is shortest, $f_i \cap f_{i+2} = \varnothing$ ($i, i+2 \in \mathbb{Z}_{r+1}$). So $S' = \{f_i \cap f_{i+1} | i \in \mathbb{Z}_{r+1}\}$ is an cyclic edge-cut. By Theorem 2.2, $|S'| \geq 5$ and hence $r \geq 4$.

If $diam(G^*) \leq 3$, then $r \leq 3$, contradicting that $r \geq 4$. The contradiction implies that ∂f is a cycle. So $F - G$ is 2-connected.

Now suppose that $diam(G^*) = 4$. Then $r = 4$ and $|S'| = 5$. Hence S' is a cyclic 5-edge-cut of F such that $e_i \in S'$ and P is a path of length 4 in G^*. □

Let F be a fullerene graph and F_1 and F_2 be two subgraphs of F. Denote the set of edges connecting vertices of F_1 with vertices of F_2 by $E(F_1, F_2)$. We use $E(e, F_2)$ instead of $E(F_1, F_2)$ if F_1 is an edge e of F. Let S be a set of edges in F. Use $F \ominus S$ to denote the subgraph arising from deleting all vertices incident with edges in S together with their incident edges. We say that S *forces* an edge e of F if one end of e is of degree 1 in $F \ominus S$.

Theorem 2.7. *Let F be a fullerene graph. Then, for every perfect matching M of F, $f(F; M) \geq 3$.*

Proof. Since any two independent edges of a fullerene are contained in a perfect matching by Theorem 2.1, it suffices to show that any pair of independent edges e_1 and e_2 of a fullerene graph F cannot determine uniquely a perfect matching; that is, $F \ominus \{e_1, e_2\}$ has at least two perfect matchings. Let $F' := F \ominus \{e_1, e_2\}$ and $e_1 = v_1v_2$, $e_2 = u_1u_2$.

Claim 1. *If F' has no vertices of degree 1, then F' is 2-connected.*

Claim 2. *If F' has a 1-degree vertex, then by repeatedly deleting the resulting 1-degree vertices and their neighbors (in any order) we arrive at a 2-connected subgraph F''.*

By Theorem 2.5, a 2-connected graph with a perfect matching has at least two distinct perfect matchings. Hence, by Claims 1 and 2, F' has at least two distinct perfect matchings. The proof is complete. □

Now we are going to prove the Claims.

Proof of Claim 1. First we prove that F' is connected. If not, then F' has two components F_1' and F_2'. Note that, since F' has no vertices of degree 1, F_i' contains a cycle, $i = 1, 2$. We may assume that $|E(e_1, F_1')| \leq |E(e_1, F_2')|$. Since $|E(e_i, F_1')| + |E(e_i, F_2')| = 4$ ($i = 1, 2$), $|E(e_1, F_1')| \leq 2$. Note that, since $\delta(F') \geq 2$, any edge-cut separating F_1' and F_2' must be cyclic. Furthermore, at least one of $E(e_1, F_1') \cup E(e_2, F_2')$ and $E(e_1, F_1') \cup E(e_2, F_1')$ is a cyclic edge-cut of size no more than 4. This contradicts $c\lambda(F) = 5$. So F' is connected.

Now we are going to prove that F' is 2-connected. Suppose not; since F' is connected, F' has a bridge $e = w_1w_2$. Removing e from F' separates F' into two components F_1' and F_2' such that $w_i \in F_i'$. Clearly, both F_1' and F_2' have cycles since $\delta(F') \geq 2$.

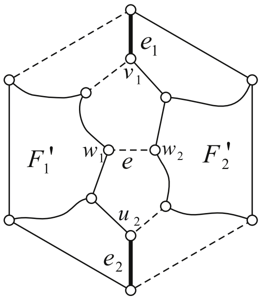

Fig. 2. Illustration for the proof of Claim 1.

Let f_1 and f_2 be two faces of F such that $e = f_1 \cap f_2$. If one face of f_1 and f_2, say f_1, contains no end of e_1 and e_2, then e cannot be a bridge of F' since e lies on the cycle bounding f_1. If one edge of e_1 and e_2, say e_1, has an end on each f_i, then let e' and e'' be the two edges of f_1 and f_2, respectively, such that they are incident with one end of e_1. Then $\{e, e', e''\}$ forms a cyclic 3-edge-cut. This contradicts $c\lambda(F) = 5$. Thus we may assume that $f_1 \cap e_1 \neq \varnothing$ and $f_2 \cap e_2 \neq \varnothing$.

If $e_2 \subset f_2$, then $S := E(e_1, F_1') \cup \{e, e_2\}$ is an edge-cut of size at most 4 such that $F - S$ has two components containing F_1' and F_2', respectively. So S is a cyclic edge-cut, a contradiction to $c\lambda(F) = 5$.

Thus we may assume that $e_1 \notin E(f_1)$, $e_2 \notin E(f_2)$, $v_1 \in f_1$ and $u_2 \in f_2$. Since both $E(e_1, F_1') \cup E(e_2, F_1') \cup \{e\}$ and $E(e_1, F_1') \cup E(e_2, F_2') \cup \{e\}$ are cyclic edge-cuts of F, $|E(e_1, F_1') \cup E(e_2, F_1')| + 1 \geq c\lambda(F) = 5$ and $|E(e_1, F_1')| + |E(e_2, F_2')| + 1 \geq c\lambda(F) = 5$. Since $|E(e_2, F_1')| + |E(e_2, F_2')| = 4$ and $|E(e_1, F_1')| \leq 2$, we have $|E(e_2, F_1')| = |E(e_2, F_2')| = 2$. Hence $|E(e_1, F_1')| + |E(e_2, F_2')| = 4$. Therefore, $S := E(e_1, F_1') \cup E(e_2, F_2') \cup \{e\}$ is a cyclic 5-edge-cut (the broken edges shown in Fig. 2).

We are going to prove that S is non-trivial. Note that, if two edges of $E(e_2, F_1')$ are incident with an end vertex of e_2, then $e_2 \in E(f_2)$. This is a direct contradiction to $e_2 \notin E(f_2)$. Similarly for the edges of $E(e_1, F_1')$.

Since F_1' is connected and does not contain vertex of degree 1, F_1' contains a cycle C and the edges in $E(e_2, F_1')$ are disjoint. Let xu_2 and yu_1 be the edges in $E(e_2, F_1')$. Let P be a path in F_1' connecting x and y. Then $C_1 = Pu_1u_2x$ is a cycle in $F_1' \cup \{e_2\} \cup E(e_2, F_1')$ which is a component of $F - S$. Clearly C and C_1 are distinct. Hence $F_1' \cup \{e_2\} \cup E(e_2, F_1')$ contains two cycles. We have a similar result for $F_2' \cup \{e_1\} \cup E(e_1, F_2')$. Thus, S is non-trivial.

Let G be the component of $F - S$ containing F_2'. Then G contains exactly five vertices of degree 2. But there are two consecutive vertices of degree 2, which are the ends of e_1. This contradicts that S is a non-trivial cyclic 5-edge-cut by Proposition 2.4. The contradiction implies that F' is 2-connected and hence completes the proof of Claim 1. □

Before proving Claim 2 we have the following properties.

Let G be a subgraph of a fullerene graph F. Let $d_G(v)$ denote the degree of a vertex of G. A vertex v of G is *unsaturated* if $d_G(v) \leq 2$.

Let G be a connected subgraph of a fullerene graph F with a cycle such that every inner face of G is also a face of F. Suppose that G contains at least one unsaturated vertex and any two unsaturated vertices which belong to a common face of F have no common neighbors in $F - G$. (*)

Note that every unsaturated vertex of G lies on its boundary ∂G since every inner face of G is also a face of F. Each face of F is bounded by a cycle.

A path connecting two unsaturated vertices w and v such that every intermediate vertex is of degree 3 is called a *saturated path*. For convenience, a face is often represented by its boundary. Note that, for a saturated path P on ∂G, there exists a face f such that $P \subset f \cap G$.

Lemma 2.8. *Let G be a connected subgraph of F satisfying the condition (*) and not isomorphic to a pentagon. Let u_0, u_1, u_2 and u_3 be four unsaturated vertices of G such that u_i and u_{i+1} $(i = 0, 1, 2)$ are connected by a saturated path P_i on the boundary of G. Then u_i and u_j $(i \neq j)$ have no common neighbors in $F - G$.*

Proof. We may assume that $i < j$.

Case 1: $j - i = 1$. Then u_i and u_{i+1} are connected by a saturated path P_i. Furthermore, u_i and u_{i+1} belong to a face f_i of F such that $P_i \subset f_i \cap G$. So u_i and u_j have no common neighbors in $F - G$ by the assumption that any two unsaturated vertices belonging to a common face of F have no common neighbors in $F - G$.

Case 2: $j - i = 2$. Without loss of generality, we may assume that $i = 0$ and $j = 2$. Suppose that $d_G(u_1) = 1$. Since F is a cubic graph, $P_0 \cap P_1$ must be an edge containing u_1. Let this edge be $e = u_1u_1'$. Suppose that $d_G(u_1) = 2$. Then there is an edge $e = u_1u_1'$, where $u_1' \in V(F) \setminus V(G)$. Let f_0 be the face in F containing P_0 and e and let f_1 be the face in F containing P_1 and e. Suppose to the contrary that u_0 and u_2 have a common neighbor w in $F - G$. By the assumption, $f_0 \neq f_1$. Note that, since F is a cubic graph, $f_0 \cap f_1 = e$. For convenience, we let $\alpha(e) = u_1$ if $d_G(u_1) = 1$ and $\alpha(e) = u_1'$ if $d_G(u_1) = 2$.

Suppose that $wu_0 \in E(f_0)$. If $wu_2 \in E(f_1)$, then w is connected to u_1 by a path P which contains neither u_0 nor u_2 because $f_0 \neq f_1$. Since u_0 and u_1 have no common neighbors in $F - G$, P is not an edge. Hence, $\{w, u_1\}$ is a 2-vertex-cut of F. This

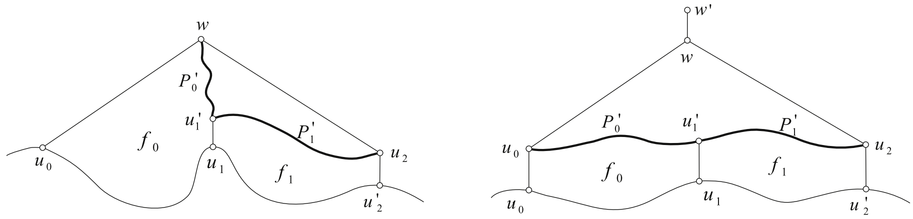

Fig. 3. Illustration for Case 2.

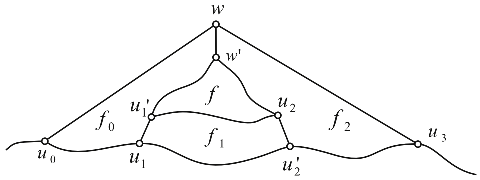

Fig. 4. Illustration for Case 3.

contradicts the 3-connectivity of F. Hence $wu_2 \notin E(f_1)$. Hence $d_G(u_2) = 1$ since $f_1 - P_1$ contains a neighbor of u_2. Let $u_2'u_2 \in E(G)$. Let $P_0' \subset f_0$ be the path connecting w and $\alpha(e)$ such that $u_0 \notin P_0'$, and let $P_1' \subset f_1$ be the path connecting $\alpha(e)$ and u_2 such that P_1' does not contain any edge of G. Then $C := P_0' \cup P_1' \cup \{u_2w\}$ is a cycle. So $S = \{u_0w, e, u_2u_2'\}$ is a cyclic 3-edge-cut since it separates F into two components: one contains C and the other contains a cycle of G (see Fig. 3 (left)). This contradicts $c\lambda(F) = 5$.

Suppose that $wu_0 \notin E(f_0)$ and $wu_2 \notin E(f_1)$. Then $d_G(u_0) = d_G(u_1) = 1$. Let $u_0'u_0 \in E(G)$. Let $P_0' \subset f_0$ be the path connecting u_0 and $\alpha(e)$ and let $P_1' \subset f_1$ be the path connecting $\alpha(e)$ and u_2 such that both of them do not contain edges of G. Then $C := P_0' \cup P_1' \cup \{u_0wu_2\}$ is a cycle. Let w' be the neighbor of w different from u_0 and u_1. Then $\{u_0u_0', e, u_2u_2', ww'\}$ contains a cyclic edge-cut which separates F into two components: one contains C and the other contains a cycle of G. This contradicts $c\lambda(F) = 5$. Hence we have that u_0 and u_2 have no common neighbors in $F - G$.

Case 3: $j-i = 3$. Similarly to Case 2, we may choose faces f_0 and f_1 such that $P_j \subset f_j \cap G$ for $j = 0, 1$ and $f_0 \cap f_1 = e_1 = u_1u_1'$. If $d_G(u_2) = 2$, then there is an edge $e_2 = u_2u_2'$ such that $u_2' \in V(F) \setminus V(G)$. Let f_2 be the face containing $\{e_2\} \cup P_2$. If $e_2 \notin E(f_1)$, then by a similar proof to that of Case 2 we will obtain that u_0 and u_2 have no common neighbors in $F - G$. So we assume that $e_2 \in E(f_1)$. Hence $f_1 \cap f_2 = e_2$.

If $d_G(u_2) = 1$, then similarly to Case 2 we have that $P_1 \cap P_2$ is an edge $e_2 = u_2u_2'$ for some u_2'. Let f_2 be the face containing P_2. Then $f_1 \cap f_2 = e_2$.

Combining the above cases, we have three distinct faces f_0, f_1 and f_2 such that $P_j \subset f_j \cap G$ and $e_i = f_{i-1} \cap f_i$ for $i = 1, 2$. Denote that $\alpha(e_i) = u_i$ if $d_G(u_i) = 1$ and $\alpha(e_i) = u_i'$ if $d_G(u_i) = 2$. Let $P_1' \subset f_1$ be a path connecting $\alpha(e_1)$ to $\alpha(e_2)$ such that P_1' contains no edges of G.

Now suppose to the contrary that w is the neighbor of u_0 and u_3 in $F - G$. Suppose that $wu_0 \in E(f_0)$. If $wu_3 \in E(f_2)$, then $f_0 \cap f_2 \neq \emptyset$. Since F is cubic, $f_0 \cap f_2$ is an edge which is ww' for some w' (see Fig. 4). If $w' \in f_1$, then $w'u_i = f_{i-1} \cap f_i$ for $i = 1, 2$. Furthermore, w' is a common neighbor of u_1 and u_2. This contradicts the assumption of G that u_1 and u_2 have no common neighbors in $F - G$. So w' lies on a face f which is different from f_i for $i = 0, 1$ and 2. Then $\{ww', e_1, e_2\}$ is a cyclic 3-edge-cut which separates F into two components, of which one contains a cycle in G and the other is f. This contradicts $c\lambda(F) = 5$.

So we assume that $wu_3 \notin E(f_2)$. That means that $d_G(u_3) = 1$. Let $u_3u_3' \in E(G)$. Let $P_2' \subset f_2$ be a path connecting $\alpha(e_2)$ and u_3 but $u_3' \notin P_2'$ and let $P_0' \subset f_0$ be a path connecting w to $\alpha(e_1)$ but $u_0 \notin P_0'$. Then $C := (\bigcup_{i=0}^{2} P_i') \cup \{u_3w\}$ is a cycle. Hence $\{u_0w, e_1, e_2, u_3u_3'\}$ is a cyclic 4-edge-cut since it separates F into two components, of which one contains the cycle C and the other contains a cycle in G. This contradicts $c\lambda(F) = 5$.

In the following, we assume that $wu_0 \notin E(f_1)$ and $wu_3 \notin E(f_2)$. Hence $d_G(u_0) = d_G(u_3) = 1$ and hence let $u_ju_j' \in E(G)$ for $j = 0, 3$. Let $P_0' \subset f_0$ be the path connecting u_0 to $\alpha(e_1)$ but $u_0' \notin P_0'$. So $C := (\bigcup_{i=0}^{2} P_i') \cup \{u_0wu_3\}$ is a cycle. Let G' be the subgraph of F consisting of C together with its interior. If w has a neighbor inside C, then $\{u_0u_0', e_1, e_2, u_3u_3'\}$ forms a cyclic 4-edge-cut of F. This contradicts $c\lambda(F) = 5$. Thus, w has a neighbor $w' \notin V(G')$. Then $S = \{u_0u_0', e_1, e_2, u_3u_3', ww'\}$ is a cyclic 5-edge-cut. Note that C is not a pentagon since each P_i' is not an edge by the assumption that any two unsaturated vertices which belong to a common face of F have no common neighbors in $F - G$. So S is non-trivial since G is also not a pentagon. However, w, u_1, u_3 are three vertices of degree two on the boundary of G' which appear consecutively. This contradicts Proposition 2.4. The contradictions imply that u_0 and u_3 have no common neighbors in $F - G$. $\square$

Suppose that G satisfies the condition (*). Let u and v be a pair of unsaturated vertices in G. Suppose that Q is a path on ∂G connecting u and v. This path is called a (u, v)-*path* on ∂G. We define the number $\ell(Q)$ to be the number of unsaturated vertices of G contained in Q minus one. The *unsaturated distance* of u and v is the minimum number of $\ell(Q)$ when Q runs through all the (u, v)-paths on ∂G.

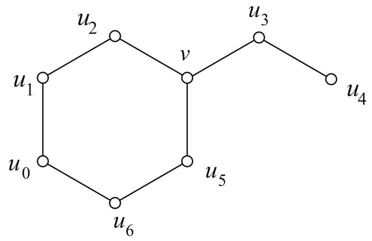

Fig. 5. An example.

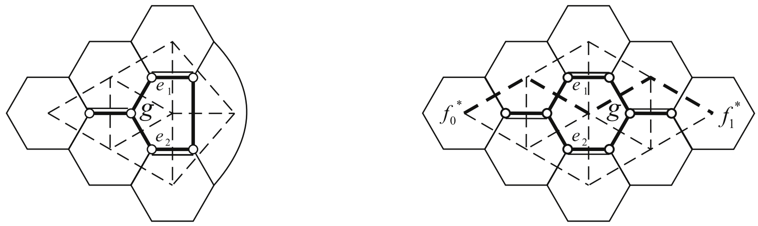

Fig. 6. F has a face g containing both e_1 and e_2.

Consider the graph given in Fig. 5. Suppose that it is a subgraph of a fullerene graph satisfying the condition (*). Then the unsaturated distance of u_1 and u_5 is 2; and that of u_0 and u_4 is 4.

From the proof of Lemma 2.8 we have the following corollary.

Corollary 2.9. *Suppose that G is a connected subgraph of F satisfying the condition (*) and not isomorphic to a pentagon. If each pair of unsaturated vertices in G is of unsaturated distance at most* 3*, then G is unforcing.*

Proof. Since any two unsaturated vertices of G have no common neighbors in $F-G$, $F-G$ does not contain any vertex of degree 1. By the assumption, every inner face of G is also a face of F. Thus, G is unforcing. □

Now, we are going to prove Claim 2. The same idea for proving Claim 2 is also used by Došlić in [5].

Proof of Claim 2. Let $G := F - F''$, where F'' is the finial subgraph obtained from F' by repeatedly deleting the resulting 1-degree vertices and their neighbors. In the following figures, we use thick lines to illustrate the graph G and broken lines to illustrate the graph G^*.

Let w be a 1-degree vertex of F'. Since F does not contain a 3-cycle, w must be adjacent with an end vertex of each e_i for $i = 1, 2$. Recall that $e_1 = v_1v_2$ and $e_2 = u_1u_2$. We may assume that wv_1 and wu_1 are edges of F. Let g_1 be the face in F containing e_1 and wv_1 and let g_2 be the face in F containing e_2 and wu_1. Since F is a cubic graph, g_1 and g_2 have at least one common edge.

Suppose that $g_1 = g_2 = g$. If g is a pentagon, then G is isomorphic to the graph shown on the left in Fig. 6. Clearly, $diam(G^*) = 3$, and by Corollary 2.9 G is unforcing. By Lemma 2.6, $F'' = F - G$ is 2-connected. Now if g is a hexagon, then G is isomorphic to the graph shown on the right in Fig. 6. Then by Corollary 2.9 G is unforcing, and clearly $diam(G^*) = 4$. If F'' is not 2-connected, then F'' has a bridge e by Lemma 2.6. Assume that $e = f_0 \cap f_1$ and P is a shortest path joining f_0^* and f_1^*. Note that G^* has only one pair of vertices f_0^* and f_1^* which has distance 4, as shown in Fig. 6. Let P be the 4-length path shown in thick broken lines in Fig. 6. Together with e, the edges corresponding to edges of P can form neither a trivial nor a non-trivial cyclic 5-edge-cut by Proposition 2.4. This contradicts Lemma 2.6.

Now we assume that $g_1 \neq g_2$. Let $e = ww'$ be the edge of F' incident with w.

Case 1: Suppose that e is the common edge of g_1 and g_2. If both g_1 and g_2 are hexagonal, then G is a tree, as shown in Fig. 7 (left). By Corollary 2.9, G is unforcing. It is easy to check that $diam(G^*) = 3$. By Lemma 2.6, F'' is 2-connected.

If exactly one of g_1 and g_2 is pentagonal, say g_1, then g_1 contains a vertex w_1 different from v_1, v_2, w and w'. Let w_1w_2 be the edge incident with w_1 in the subgraph $F \ominus \{e_1, e_2, e\}$. Then G is induced by $V(g_1) \cup \{w_2, u_1, u_2\}$ in F, as shown in Fig. 7 (middle). By Corollary 2.9, G is unforcing. It is easy to see that $diam(G^*) = 3$. By Lemma 2.6, F'' is 2-connected.

If both g_1 and g_2 are pentagonal, then $F \ominus \{e_1, e_2, e, w_1w_2\}$ has one 1-degree vertex w_3. Let w_3w_4 be the edge forced by e and e_2 and let g be the face of F containing w_1w_2 and w_3w_4. If g is pentagonal, then $G := g \cup g_1 \cup g_2$ and $diam(G^*) = 3$. By Corollary 2.9, G is unforcing. Hence F'' is 2-connected by Lemma 2.6. If g is hexagonal, then g has a 1-degree vertex w_5 of $F \ominus \{e_1, e_2, e, w_1w_2, w_3w_4\}$. Let w_5w_6 be the edge in $F \ominus \{e, e_1, e_2, e, w_1w_2, w_3w_4\}$. Hence G consists of $g \cup g_1 \cup g_2$ together with w_5w_6, as shown in Fig. 7 (right). It is easy to see that $diam(G^*) = 4$ and G^* has only one pair of vertices f_0^* and f_1^* with distance 4. Let P be a shortest path connecting f_0^* and f_1^* illustrated by the thick broken lines. If F'' has a bridge e', then $e' = f_0 \cap f_1$, where f_0 and f_1 are faces of F corresponding to vertices f_0^* and f_1^*. By Lemma 2.6, together with e', the edges corresponding to edges of P form a cyclic 5-edge-cut S. Clearly, S is non-trivial. This contradicts $g_1 \cap g \in S$ by Proposition 2.4. Hence F'' is 2-connected by Lemma 2.6.

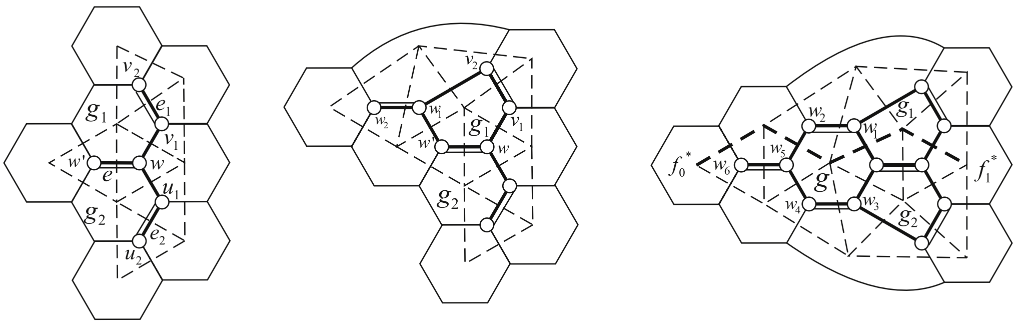

Fig. 7. Illustration for the proof of Case 1.

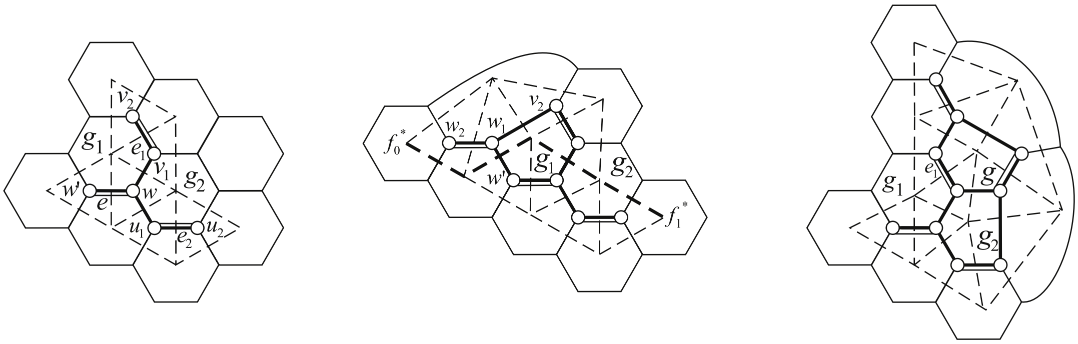

Fig. 8. Illustration for the proof of Subcase 2.1.

Case 2: Suppose that e is not the common edge of g_1 and g_2. We may assume that e lies on the boundary of g_1. If both g_1 and g_2 are hexagonal, then G induced by $\{v_1, v_2, u_1, u_2, w, w'\}$ is a tree (see Fig. 8 (left)) and $diam(G^*) = 3$. By Corollary 2.9, G is unforcing. By Lemma 2.6, we immediately have that $F'' = F - G$ is 2-connected. So, in the following, we assume that at least one of g_1 and g_2 is pentagonal.

Subcase 2.1: Exactly one of g_1 and g_2 is pentagonal. Suppose that g_1 is pentagonal and g_2 is hexagonal, then g_1 contains a 1-degree vertex w_1 in $F \ominus \{e_1, e_2, e\}$. Let w_1w_2 be the edge in $F \ominus \{e_1, e_2, e\}$. Then G has only one inner face g_1, as shown in Fig. 8 (middle). By Corollary 2.9, G is unforcing. It is easy to check that $diam(G^*) = 4$ and G^* has only one pair of vertices, say f_0^* and f_1^*, with distance 4. Let P be the thick broken lines connecting f_0^* and f_1^*. If F'' has a bridge e', then $e' = f_0 \cap f_1$, where f_0 and f_1 are faces of F corresponding to f_0^* and f_1^*, respectively. By Lemma 2.6, together with e', the edges corresponding to edges of P form a cyclic 5-edge-cut S. Since S contains edges in the pentagon g_1, S cannot be non-trivial by Proposition 2.4. It also cannot be trivial since g_2 is a hexagon. So F'' has no bridges. That is, F'' is 2-connected by Lemma 2.6.

Suppose that g_2 is pentagonal and g_1 is hexagonal. Let g be the face of F adjacent with g_1 such that $g_1 \cap g = e_1$, as shown in Fig. 8 (right). Whether g is hexagonal or not, we always have that G is unforcing and $diam(G^*) = 3$. Hence F'' is 2-connected by Lemma 2.6.

Subcase 2.2: Both g_1 and g_2 are pentagonal. Let $w_1 \in g_1$ and $w_2 \in g_2$ be the 1-degree vertices of $F \ominus \{e, e_1, e_2\}$. Let w_1w_3 and w_2w_4 be the two edges in $F \ominus \{e_1, e_2, e\}$. Let g be the face of F such that $g \cap g_1 = e_1$. If g is a hexagon, then G consists of g_1 and g_2 together with two edges w_1w_3 and w_2w_4 (see Fig. 9 (left)). By Corollary 2.9, G is unforcing. It is easy to check that $diam(G^*) = 3$. By Lemma 2.6, $F'' = F - G$ is 2-connected. If g is a pentagon, then g contains a 1-degree vertex w_5 in $F \ominus \{e_1, e_2, e, w_1w_2, w_3w_4\}$. Let w_7 be adjacent with w_5 in $F \ominus \{e_1, e_2, e, w_1w_2, w_3w_4\}$. Let g' be the face adjacent with both g and g_1. Then $w_1w_3 \subset g'$.

If g' is a pentagon, then $F \ominus \{e_1, e_2, e, w_1w_3, w_2w_4, w_5w_7\}$ has no 1-degree vertices (see Fig. 9 (middle)). Hence $G = g \cup g' \cup g_1 \cup g_2$ and $diam(G^*) = 3$. By Corollary 2.9, G is unforcing. By Lemma 2.6, F'' is 2-connected.

If g' is a hexagon, then $F \ominus \{e_1, e_2, e, w_1w_3, w_2w_4, w_5w_7\}$ has a 1-degree vertex w_6 (see Fig. 9 (right)). Let w_8 be adjacent with w_6 in $F \ominus \{e_1, e_2, e, w_1w_3, w_2w_4, w_5w_7\}$. Then $F \ominus \{e_1, e_2, e, w_1w_3, w_2w_4, w_5w_7, w_6w_8\}$ has no 1-degree vertices. So G consists of g_1, g_2, g, g' together with the edge w_6w_8 and $diam(G^*) = 4$. By Corollary 2.9, G is unforcing. Note that G^* has two pairs of vertices with distance 4. One is f_0^* and g_2^*, the vertex of G^* corresponding to the face g_2. Another is f_0^* and f_1^*. Let f_0 and f_1 be two faces of F corresponding to f_0^* and f_1^*, respectively. Clearly, f_0 cannot be adjacent with g_2 in F. Suppose that $F'' = F - G$ is not 2-connected. By Lemma 2.6, $e' = f_0 \cap f_1$ is a bridge of F''. Let P be the shortest path joining f_0^* and f_1^*, as shown by broken dashed lines in Fig. 9 (right). Then e' and the four edges corresponding to edges of P form a cyclic 5-edge-cut S of F by Lemma 2.6. Clearly, S is non-trivial. By Theorem 2.3, S separates F into two subgraphs, each of which contains a pentacap. Let F_1 be the subgraph containing a pentacap H which contains the pentagons g_1, g_2 and g which are adjacent with each other. Since g' is a hexagon and is adjacent with g_1 and g, g_2 should be the pentagon of H adjacent with

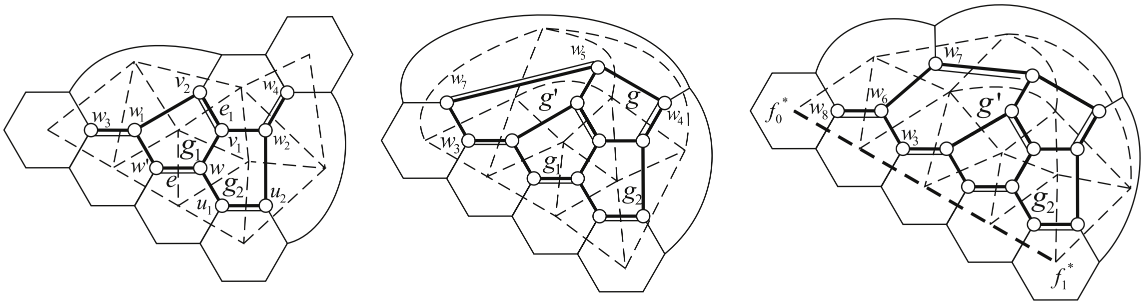

Fig. 9. Illustration for the proof of Subcase 2.2.

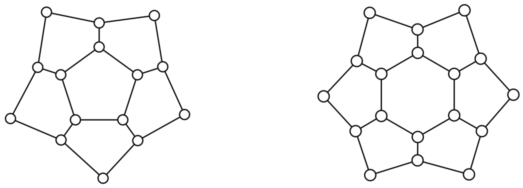

Fig. 10. A pentacap (left) and a hexacap (right).

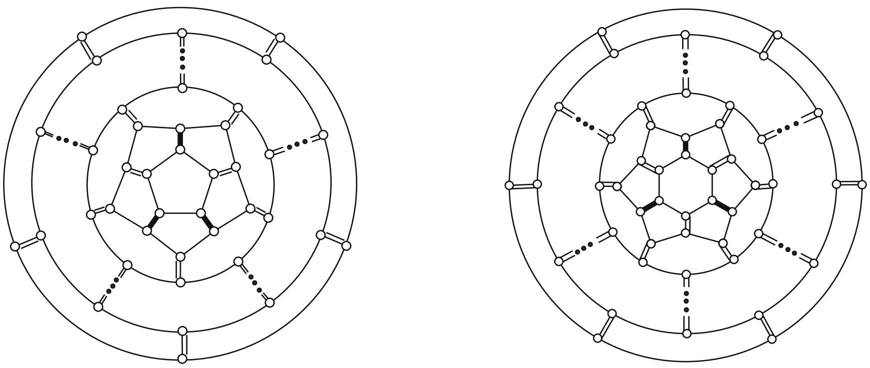

Fig. 11. Fullerene graphs F_P^i (left) and F_H^i (right).

all the other five pentagons of H. That means that all faces adjacent with g_2 are pentagons, and so is f_1. However, $S \cap f_1 \neq \emptyset$, a contradiction to Proposition 2.4. So $F'' = F - G$ is 2-connected. This completes the proof of Claim 2. $\square$

So our main result Theorem 1.1 follows directly from Theorem 2.7.

3. Sharpness of the bound

In this section, we will construct infinitely many fullerene graphs attaining the lower bound given in Theorem 1.1.

A *hexacap* H is a graph consisting of a hexagon together with six adjacent pentagons, as shown in Fig. 10. Let G_0 be a hexacap and let $G_{i+1} = G_i \cup \left(\bigcup_{j=0}^{5} h_j\right)$, where h_j is a hexagon such that $h_j \cap G_i$ is a path of length 2, and h_j and h_{j+1} have exactly one common edge ($j \in \mathbb{Z}_6$). We add a hexacap to G_i along their boundaries by identifying the 2-degree (resp. 3-degree) vertices of G_i with the 3-degree (resp. 2-degree) vertices of the hexacap. Finally we get a fullerene graph, denoted by F_H^i. Note that F_H^0 is isomorphic to F_{24}, which is the unique fullerene graph with 24 vertices [8].

Similarly, let K_0 be a pentacap and let $K_{i+1} = K_i \cup \left(\bigcup_{j=0}^{4} h_j\right)$, where h_j is a hexagon such that $h_j \cap K_i$ is a path of length 2, and h_j and h_{j+1} have exactly one common edge ($j \in \mathbb{Z}_5$). Then we add a hexacap to K_i along their boundaries by identifying the 3-degree (resp. 2-degree) vertices of K_i with the 2-degree (resp. 3-degree) vertices of the pentacap. Finally we get a fullerene graph, denoted by F_P^i. Note that F_P^0 is isomorphic to the dodecahedron F_{20}.

Theorem 3.1. *Let F be a fullerene graph isomorphic to F_H^i or F_P^i for some $i \geq 0$. Then $f(F) = 3$.*

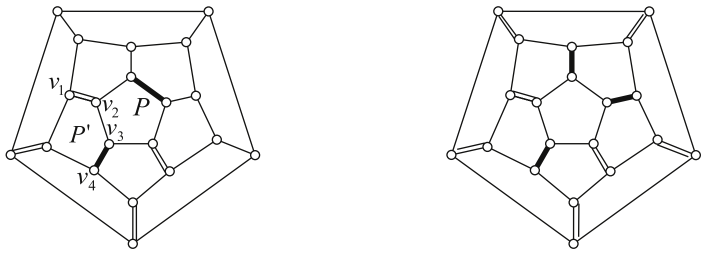

Fig. 12. The dodecahedron F_{20}.

Proof. Let F be a fullerene graph isomorphic to F_H^i or F_P^i. Let G_0 be a pentacap (resp. hexacap) of F. Then F can be constructed as mentioned above. Note that there are 5 (resp. 6) edges with exactly one end on the central pentagon (hexagon) of G_0. Denote the set of these edges by S_0. Let e_1, e_2, e_3 be three of them, which are shown as the thick edges in Fig. 11, and let $S = S_0 \setminus \{e_1, e_2, e_3\}$. If F has a perfect matching M such that $e_1, e_2, e_3 \in M$, then $S \subset M$. Furthermore, $E(G_j, F - G_j) \subset M$, where $j = 0, 1, \ldots, i$. Immediately, $M = \{e_1, e_2, e_3\} \cup S \cup \left(\bigcup_{j=0}^{i} E(G_j, F - G_j)\right)$ is uniquely determined by $\{e_1, e_2, e_3\}$. So $f(F) \le f(F; M) \le 3$. By Theorem 2.7, we have $f(F) = 3$. □

By Theorem 2.3, a fullerene graph F with a nontrivial cyclic 5-edge-cut is isomorphic to F_P^i for some positive integer i. So we have the following result.

Corollary 3.2. *Let F be a fullerene graph with a nontrivial cyclic 5-edge-cut. Then $f(F) = 3$.*

4. Remarks and open problems

As we have already shown, there are infinitely many fullerene graphs with minimum forcing number 3. It is interesting to characterize all such fullerene graphs.

Problem 4.1. Determine all fullerene graphs for which the minimum forcing number is equal to 3.

We discuss only the lower bound for the forcing number of perfect matchings of fullerene graphs here. The sharp upper bound for the forcing number of perfect matchings of fullerene graphs is also interesting.

Problem 4.2. Give a sharp upper bound for the forcing number of perfect matchings of fullerene graphs.

Proposition 4.3. *Let M be a perfect matching of the dodecahedron F_{20}. Then $f(M, F_{20}) = 3$.*

Proof. Let M be a perfect matching of F_{20}. Suppose that F_{20} has a pentagon P' containing two edges in M. Let $v_1v_2, v_3v_4 \in P' \cap M$ and assume that v_2 is adjacent with v_3. Let P be the pentagon of F_{20} such that $v_2v_3 = P \cap P'$. Then $|P \cap M| \le 1$ (see Fig. 12 (left)). Consequently, F_{20} has a pentagon, say P, containing at most one edge in M.

If $|M \cap P| = 1$, then the two thick edges determine a matching M', as shown in Fig. 12 (left). Then $F_{20} \ominus M'$ consists of two adjacent pentagons. It is easy to see that $F_{20} \ominus M'$ has only two perfect matchings with the forcing number 1. So $f(F_{20}, M) = 3$. Suppose that $|M \cap P| = 0$. Then the three thick edges determine the unique perfect matching of F_{20}, as shown in Fig. 12 (right). So $f(M, F_{20}) = 3$. □

By Proposition 4.3, we propose the following open problem.

Problem 4.4. Determine all fullerene graphs of which all perfect matchings have the same forcing number.

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